SOLUTIONS FOR PROBLEMS # 38 and #39 in Sec 11.3

SEC 11.3, #38 Solution (page 1)

FIND the sum s of the series $\sum_{n=1}^{\infty} n e^{-2n}$ correct to four decimal places.

Sol'n: We first must show that the series $\sum_{n=1}^{\infty} n e^{-2n}$ converges in the first place.

Let $f(x) = x e^{-2x}$.

The function f is continuous on $[1,\infty)$, f is positive on $[1,\infty)$ and $f(n) = n e^{-2n}$ for each integer $n \geq 1$.

We show that f is decreasing on $[1,\infty)$ by showing that $f'(x) < 0$ for all x such that $x \geq 1$.

$f(x) = x e^{-2x}$, so, $f'(x) = 1 \cdot e^{-2x} + x(-2e^{-2x})$.

So, $f'(x) = (1-2x)e^{-2x}$. For $x \geq 1$, $1-2x < 0$ and $e^{-2x} > 0$.

So $f'(x) = (\text{NEG. \#})(\text{POS \#}) < 0$ on $[1,\infty)$.

$\therefore$ $f(x)$ is decreasing on $[1,\infty)$.

Therefore, the Integral Test applies.

Let n be any integer such that $n \geq 1$.

$$\int_n^{\infty} x e^{-2x}\,dx = \lim_{t\to\infty} \int_n^t x e^{-2x}\,dx$$

Using "Parts":
$u = x$, $dv = e^{-2x}dx$
$du = 1 \cdot dx$, $v = -\frac{1}{2}e^{-2x}$

$$= \lim_{t\to\infty}\left[\left(-\frac{1}{2}x e^{-2x}\Big|_n^t\right) - \int_n^t \left(-\frac{1}{2}e^{-2x}\right)dx\right]$$

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$$= \lim_{t\to\infty}\left[\left(\left(-\tfrac{1}{2}te^{-2t}\right)-\left(-\tfrac{1}{2}ne^{-2n}\right)\right)+\tfrac{1}{2}\left(\int_n^t e^{-2x}\,dx\right)\right]$$

$$= \lim_{t\to\infty}\left[\left(-\tfrac{1}{2}te^{-2t}+\tfrac{1}{2}ne^{-2n}\right)+\tfrac{1}{2}\left(-\tfrac{1}{2}e^{-2x}\Big|_n^t\right)\right]$$

$$= \lim_{t\to\infty}\left[\left(-\tfrac{1}{2}te^{-2t}+\tfrac{1}{2}ne^{-2n}\right)+\tfrac{1}{2}\left(-\tfrac{1}{2}e^{-2t}-\left(-\tfrac{1}{2}e^{-2n}\right)\right)\right]$$

$$= \lim_{t\to\infty}\left[\frac{-t}{2e^{2t}}+\frac{n}{2e^{2n}}-\frac{1}{4e^{2t}}+\frac{1}{4e^{2n}}\right]$$

$$= \lim_{t\to\infty}\left[\left(\frac{-t}{2e^{2t}}+\frac{-1}{4e^{2t}}\right)+\overbrace{\left(\frac{n}{2e^{2n}}+\frac{1}{4e^{2n}}\right)}^{=\frac{2n+1}{4e^{2n}}}\right]$$

$$= \left(\lim_{t\to\infty}\frac{-t}{2e^{2t}}\right)+\left(\lim_{t\to\infty}\frac{-1}{4e^{2t}}\right)+\lim_{t\to\infty}\left(\frac{2n+1}{4e^{2n}}\right)$$

Form: $\frac{-\infty}{\infty}$ $\quad$ -1, ∞ $\quad$ Constant!

$$= \lim_{t\to\infty}\frac{-1}{4e^{2t}}+0+\frac{2n+1}{4e^{2n}} = 0+0+\frac{2n+1}{4e^{2n}} = \frac{2n+1}{4e^{2n}}.$$

-1, ∞

So, For each $n \geq 1$, $\displaystyle\int_n^\infty xe^{-2x}\,dx = \frac{2n+1}{4e^{2n}}$.

In particular, when $n=1$, this analysis shows that the improper integral $\int_1^\infty xe^{-2x}\,dx$ converges (to $\frac{3}{4e^2}$)

and so,

the series $\sum_{n=1}^{\infty} ne^{-2n}$ converges (to sum s).

Now, let n be any integer such that $n \geq 1$.

We seek a formula for the level of error in $s_n \approx s$. ("using s_n to approx. s")

By the formula in Box $\boxed{2}$ on page 723 of the textbook (8e)

$$(\text{The error in } s_n \approx s) = R_n \leq \int_n^\infty xe^{-2x}\,dx .$$

So, a level of error in $s_n \approx s$ is $\int_n^\infty xe^{-2x}\,dx$.

By the work above, we know that $\int_n^\infty xe^{-2x}\,dx = \frac{2n+1}{4e^{2n}}$

So, we need to determine a value for n such that

$$\int_n^\infty xe^{-2x}\,dx = \frac{2n+1}{4e^{2n}} \lesssim 0.00005 .$$

Observing the table of values on the next page, we see that, for $n=5$, $\frac{(2n+1)}{4e^{2n}} = 0.000124849807$.

So, for $n=5$, $\frac{(2n+1)}{4e^{2n}} \leq 0.0005$, but $\frac{(2n+1)}{4e^{2n}} \not\leq 0.00005$.

$$n, \quad \frac{(2n+1)}{4e^{2n}}, \quad a_n = n e^{-2n}, \quad S_n = \sum_{k=1}^{n} k e^{-2k}$$

n	(2*n + 1) / (4*EXP(2*n))	n	a-n = n/(exp(2*n))	n	s-n
1	0.101501462427	1	0.135335283237	1	0.1353352832
2	0.022894548611	2	0.036631277777	2	0.1719665610
3	0.004337816309	3	0.007436256530	3	0.1794028175
4	0.000754790913	4	0.001341850512	4	0.1807446681
5	0.000124849807	5	0.000226999649	5	0.1809716677
6	0.000019968690	6	0.000036865274	6	0.1810085330
7	0.000003118233	7	0.000005820701	7	0.1810143537
8	0.000000478274	8	0.000000900281	8	0.1810152540
9	0.000000072342	9	0.000000137070	9	0.1810153910
10	0.000000010821	10	0.000000020612	10	0.1810154116
11	0.000000001604	11	0.000000003068	11	0.1810154147
12	0.000000000236	12	0.000000000453	12	0.1810154152

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However, for $n=6$, $\dfrac{(2n+1)}{4e^{2n}} = 0.0000199668690$,

and so, for $n=6$, $\dfrac{(2n+1)}{4e^{2n}} \leq 0.00005$. $\left[\text{Note: Here, } \dfrac{(2n+1)}{4e^{2n}} = \dfrac{13}{4e^{12}}\right]$

Thus, the error $|s-s_6| \leq 0.00005$ because

the error $|s-s_6| = R_6 \leq \displaystyle\int_6^{\infty} xe^{-2x}\,dx = \dfrac{13}{4e^{12}} \leq 0.00005$.

Thus, the partial sum s_6 (as an approximation of the series sum s) is correct to 4 decimal places.

$s_6 \approx 0.1810$ (ROUNDED to 4 decimal places).

11.3, #39 Solution (page 1)

Find the sum s of the series $\sum_{n=1}^{\infty} (2n+1)^{-6}$ correct to 5 decimal places.

Solution: We first must show that the series $\sum_{n=1}^{\infty} (2n+1)^{-6}$ converges in the first place.

Let $f(x) = (2x+1)^{-6} = \frac{1}{(2x+1)^6}$.

The function f is continuous on $[1,\infty)$ since the only value of x at which f is not continuous is $x = -\frac{1}{2}$.

The function f is positive on $[1,\infty)$ and

$$f(n) = (2n+1)^{-6} \text{ for each } n \geq 1.$$

We show that f is decreasing on $[1,\infty)$ by showing that $f'(x) < 0$ for all x such that $x \geq 1$.

$f(x) = (2x+1)^{-6}$; so, $f'(x) = (-6(2x+1)^{-7})(2)$.

So, $f'(x) = \frac{-12}{(2x+1)^7} = \frac{NEG}{POS} < 0$ on $[1,\infty)$

Therefore, the Integral Test applies.

Let n be any integer such that $n \geq 1$.

$$\int_n^{\infty} (2x+1)^{-6}\,dx = \lim_{t\to\infty} \int_n^t (2x+1)^{-6}\,dx$$

$$= \lim_{t\to\infty} \frac{1}{2}\int_{(2n+1)}^{(2t+1)} u^{-6}\,du$$

Using "u-subst'n";
$u = 2x+1$
$du = 2dx$, $dx = \frac{1}{2}du$
When $x = n$, $u = 2n+1$
When $x = t$, $u = 2t+1$

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$$= \frac{1}{2} \lim_{t \to \infty} \left(\int_{(2n+1)}^{(2t+1)} u^{-6} \, du \right)$$

$$= \frac{1}{2} \lim_{t \to \infty} \left(-\frac{1}{5} u^{-5} \Big|_{(2n+1)}^{(2t+1)} \right) = \frac{1}{2} \lim_{t \to \infty} \left(\frac{-1}{5u^5} \Big|_{(2n+1)}^{(2t+1)} \right)$$

$$= \frac{1}{2} \lim_{t \to \infty} \left(\frac{-1}{5(2t+1)^5} - \left(\frac{-1}{5(2n+1)^5} \right) \right)$$

$$= \frac{1}{2} \lim_{t \to \infty} \left(\frac{\overset{\nearrow -1}{-1}}{\underset{\searrow \infty}{5(2t+1)^5}} + \frac{1}{5(2n+1)^5} \right)$$ Constant!

$$= \frac{1}{2} \left(0 + \frac{1}{5(2n+1)^5} \right) = \frac{1}{10(2n+1)^5}$$

So, For each $n \geq 1$, $\int_n^{\infty} (2x+1)^{-6} \, dx = \frac{1}{10(2n+1)^5}$.

In particular, when $n = 1$, this analysis shows that the improper integral $\int_1^{\infty} (2x+1)^{-6} \, dx$ converges $\left(\text{to } \frac{1}{10(3^5)} \right)$.

and so, the series $\sum_{n=1}^{\infty} (2n+1)^{-6}$ converges (to sum S)

Now, let n be any integer such that $n \geq 1$.

We seek a formula for the level of error in $S_n \approx S$.

("using S_n as an approximation of S")

From the work above, we know that

$$\int_n^\infty (2x+1)^{-6}\,dx = \frac{1}{10(2n+1)^5}.$$

By the formula in Box $\boxed{2}$ on page 723 of the textbook (8e),

$$(\text{the error in } S_n \approx S) = R_n \leq \int_n^\infty (2x+1)^{-6}\,dx.$$

So, a level of error in $S_n \approx S$ is $\int_n^\infty (2x+1)^{-6}\,dx$.

By the work above, we know that $\int_n^\infty (2x+1)^{-6}\,dx = \frac{1}{10(2n+1)^5}$.

We need to determine a value for n such that

$$\int_n^\infty (2x+1)^{-6}\,dx = \frac{1}{10(2n+1)^5} \leq 0.000005$$

Observing the table of values on the next page,

we see that for $n=3$, $\frac{1}{10(2n+1)^5} = 0.0000005949902$

So, for $n=3$, $\frac{1}{10(2n+1)^5} \not\leq 0.000005$

However, for $n=4$, $\frac{1}{10(2n+1)^5} = 0.000001693509.$

$$\frac{1}{10(2n+1)^5} \qquad a_n = (2n+1)^{-6} \qquad s_n = \sum_{k=1}^{n} (2k+1)^{-6}$$

n	1 / (10 [(2n+1)^5])	n	a-n = (2n+1)^(-6)	n	s-n
1	0.000411522634	1	0.001371742112	1	0.0013717421
2	0.000032000000	2	0.000064000000	2	0.0014357421
3	0.000005949902	3	0.000008499860	3	0.0014442420
4	0.000001693509	4	0.000001881676	4	0.0014461236
5	0.000000620921	5	0.000000564474	5	0.0014466881
6	0.000000269329	6	0.000000207176	6	0.0014468953
7	0.000000131687	7	0.000000087791	7	0.0014469831
8	0.000000070430	8	0.000000041429	8	0.0014470245
9	0.000000040386	9	0.000000021256	9	0.0014470458
10	0.000000024485	10	0.000000011660	10	0.0014470574
11	0.000000015537	11	0.000000006755	11	0.0014470642
12	0.000000010240	12	0.000000004096	12	0.0014470683

So, for $n=4$, $\frac{1}{10(2n+1)^5} \leq 0.000005$.

[NOTE: Here, $\frac{1}{10(2n+1)^5} = \frac{1}{10(9^5)}$]

Thus, the error $|s - s_4| \leq 0.000005$ because the error $|s - s_4| = R_4 \leq \int_4^\infty (2x+1)^{-6}\,dx = \frac{1}{10(9^5)}$

and so, $R_4 \leq \frac{1}{10(9^5)} \leq 0.000005$

Thus, the partial sum s_4 (as an approximation of the series sum s) is correct to 5 decimal places.

$s_4 \approx 0.00145$ (Rounded to 5 decimal places).